If there is no deformation anisotropy, i.e., $\varepsilon_{\alpha\beta} = 0$ then the coordinate axes $x_1, x_2$ should be selected so that one of them ( $x_2$, say) coincides with the projection of $\mathbf{l}$ on the plane of the wave front. Then $\alpha_1 = 0, s = 0, g_* = {}^1/_4 m\alpha_2^2$. As is shown in /1, 2/, in order to be able to describe the behaviour of shocks in the whole $uv$ plane, i.e., to have the complete shock adiabatic passing through the point $A\ (U, V)$ corresponding to the state before the jump (Fig.2), the anisotropy parameter $g_*$ should be small, of the order of $R^2 \sim \varepsilon^2$, where $R$ is the radius of the circle passing through the origin on which $S = \text{const}$. In this case $R^2 = U^2 + (V - \omega\alpha_2\alpha_3)^2$. In order that $g_* \sim R^2$, it is either necessary to have a sufficiently small anisotropy such that $\alpha_2 m^{1/2} \sim \varepsilon$ (at least along the $x_2$ axis), or the quantity $\alpha_2$ is small because the direction of wave propagation (the $x$ axis) is close to $\mathbf{l}$.

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# TWO APPROACHES TO THE INVESTIGATION OF ANTIPLANE DEFORMATION OF AN ISOTROPIC SOLID WITH A THIN ELASTIC INCLUSION*

V.K. OPANASOVICH

An approach is proposed to the investigation of the state of stress and strain of a piecewise-homogeneous plane consisting of a matrix and a thin tunnel-like rectangular inclusion with rounded-off corners under the assumption that such a composite body is under antiplane deformation conditions. A numerical comparison is made of the results obtained in this paper and on the basis of an approximate model /1/. It is shown that they agree satisfactorily at sufficiently large distances from the vertices of the inclusion, when the inclusion is more pliable than the matrix.

1. We assume that ideal mechanical contact conditions are satisfied on the material interfacial line $L$. We select an $OxyZ$ system of Cartesian coordinates with origin at the centre of a rectangular inclusion and the $OZ$ axis directed along the axis of body deformation. We know that the function reflecting the unit circle $\gamma$ on the contour $L$ has the form /2, 3/

$$z = x + iy = \omega(\sigma) = R\left(\sigma + \sum_{k=1}^{n} c_k\sigma^{-k}\right), \quad \sigma \in \gamma \tag{1.1}$$
$$R = \left(1 + \sum_{k=1}^{n} c_k\right)^{-1}$$

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According to the formulation of the problem, the conditions

$$\tau = \tau_0, \quad \partial w_0/\partial s = \partial w/\partial s \tag{1.2}$$

should be satisfied on the line $L$.

Here $\tau$ is the stress vector on the material interfacial line, $w$ is the non-zero displacement vector component, $s$ is the arc coordinate of the line $L$, and the zero subscript denotes that the corresponding quantity is referred to the inclusion.

We will introduce the stress function $F(z)$ into our consideration /4/. Then the relationships

$$\begin{aligned} &\Phi(\sigma) - \sigma^{-2}\overline{\Phi(\sigma)} = -2i\mu\sigma^{-1}|\omega'(\sigma)|\partial w/\partial s \\ &\Phi(\sigma) + \sigma^{-2}\overline{\Phi(\sigma)} = 2\tau\sigma^{-1}|\omega'(\sigma)|; \quad \sigma \in \gamma \\ &\Phi(\sigma) = F(\omega(\sigma))\,\omega'(\sigma) \end{aligned} \tag{1.3}$$

can be written.

For simplicity we will assume that the function $F(z)$ has the following form for large z:

$$F(z) = -i\tau_l + O(z^{-2}) \tag{1.4}$$

($\tau_{yz}^{\infty} = \tau_l$ is the tangential stress component at infinity).

Taking the relationships (1.3) into account, the boundary conditions (1.2) can then be written in the form

$$\begin{aligned} &\mu_0[\Phi(\sigma) - \sigma^{-2}\overline{\Phi(\sigma)}] = \mu[\Phi_0(\sigma) - \sigma^{-2}\overline{\Phi_0(\sigma)}] \\ &\Phi(\sigma) + \sigma^{-2}\overline{\Phi(\sigma)} = \Phi_0(\sigma) + \sigma^{-2}\overline{\Phi_0(\sigma)}; \quad \sigma \in \gamma \end{aligned} \tag{1.5}$$

We will seek the function $\Phi_0(\eta)$ in the form

$$\Phi_0(\eta) = i\sum_{k=-N_1}^{N} A_k\eta^k \tag{1.6}$$

where $A_k$ are unknown coefficients which we find on the basis of a system of linear algebraic equations obtained from the boundary conditions of the problem

$$\begin{aligned} &\sum_{n=0}^{N+2} A_{-n}(D_{k,\,-1+n} + \varkappa D_{k,\,1-n}) = \frac{2\beta R\tau_l}{\beta+1} D_{k,\,-1}, \quad k = 1, 2, \ldots, N+1 \\ &A_{-1} = 0, \quad A_0 = {}^1/_2\,(\beta+1)\,R\tau_l\,(\varkappa^2 - 1) + \varkappa A_{-2} \\ &A_i = \varkappa A_{-(i+2)}, \quad i = 1, 2, \ldots, N; \quad D_{0,0} = 1, \quad D_{0,j} = 0, \quad j \neq 0 \\ &k \geqslant 1, \quad D_{k,j} = R\Big(D_{k-1,\,j-1} + \sum_{\rho=1}^{n} c_\rho D_{k-1,\,j+\rho}\Big) \\ &j = -nk, \quad -nk+1, \ldots, k \\ &D_{k,j} = 0, \quad j < -nk \vee j > k; \quad \varkappa = (\beta-1)/(\beta+1), \quad \beta = \mu_0/\mu \end{aligned}$$

Taking account of relationships (1.6) and (1.4) we find on the basis of (1.5)

$$\Phi(\eta) = iR\tau_l(x\eta^{-2} - 1) + \frac{2i}{\beta+1}\sum_{k=2}^{N+2} A_{-k}\eta^{-k} \tag{1.7}$$

We note that as $\mu_0 \to 0$ we obtain an expression for the function $\Phi(\eta)$ from (1.7) for a body with an unloaded hole, as $\mu_0 \to \infty$ for a body with a stiff inclusion, and for $\mu_0 = \mu$ the solution of the problem for a homogeneous body. If we set $c_1 = m$ and $n = 1$ $(0 \leqslant m < 1)$, we arrive at a solution of the problem for a body with an elliptical inclusion that agrees with the solution obtained by other means /6/.

Using the formula $\tau_{xz} - i\tau_{yz} = \Phi(\eta)/\omega'(\eta)$ and taking (1.7) into account we arrive at the following expression to determine the stress in the matrix at a points on the real axis:

$$\begin{aligned} &x = R\Big(r + \sum_{k=1}^{n} c_k r^{-k}\Big), \quad r \geqslant 1, \quad \tau_{xz} = 0 \\ &\tau_{yz} = -\frac{1}{\omega'(r)}\Big[R(\varkappa r^{-2} - 1)\tau_l + \frac{2}{1+\beta}\sum_{k=2}^{M} A_{-k}r^{-k}\Big] \end{aligned}$$

We will represent the stress function $F_0(z)$ for the inclusion in the form

$$F_0(z) = i\sum_{k=1}^{N+1} B_k z^{k-1} \tag{1.8}$$

Taking (1.6) into account we will have the following relationships to determine the coefficients $B_k$:

$$B_n = \sum_{k=n-1}^{N} A_k \gamma_{n,\,k+1}$$

$$\gamma_{1,1} = \frac{1}{R}, \quad \gamma_{1,2} = 0, \quad \gamma_{1,k} = -\sum_{j=1}^{k-2} \gamma_{1,j} c_{k-j-1}$$

$$k = 3, 4, \ldots, N+1$$

$$\gamma_{k+1,m} = \sum_{i=1}^{m-k} \gamma_{k,m-i} \gamma_{1,i}; \quad m = k+1, k+2, \ldots, N+1;$$

$$k = 1, 2, \ldots, N$$

$$c_i = 0, \quad i > n$$

A numerical experiment showed that the functions (1.7) and (1.8) satisfy the boundary conditions (1.2), dependent on the parameter $N$, quite accurately.

Since $\tau_{xz} - i\tau_{yz} = F(z)$, then by using (1.8) we arrive at the following expressions to determine the state of stress in the inclusion on the real axis

$$\tau_{xz} = 0, \quad \tau_{yz} = -\sum_{k=1}^{N+1} B_k x^{k+1}$$

2. For a rectangular thin-walled inclusion of constant thickness $2h$ and length 2 the function $F(z)$ has the form /1/

$$F(z) = -i\tau_l + \frac{ih\mu}{2\mu_0} \frac{\tau_l}{\sqrt{z^2-1}} \sum_{k=1}^{M} \frac{X_k}{(z+\sqrt{z^2-1})^{2k-1}}$$

where $X_k = \pi c Y_k$ and $Y_k$ is the solution of the system of linear algebraic equations ($\delta_{1n}$ is the Kronecker delta)

$$\frac{h\mu\pi}{2\mu_0} Y_n + \sum_{k=1}^{M} \frac{Y_k}{2k-1} H(2k-1, 2n-1) = \delta_{1n}, \quad n = 1, 2, \ldots, M$$

$$H(m, n) = \frac{1}{(m+n)^2 - 1} - \frac{1}{(m-n)^2 - 1}$$

We select the constant $c$ twice: as in /1/

$$c = \begin{cases} \beta - 1, & \beta \leqslant 1 \\ 0, & \beta > 1 \end{cases} \tag{2.1}$$

and on the basis of the method proposed in /7/

$$c = (\beta - 1)\left[1 + \frac{\pi}{2} \sum_{k=1}^{M} \frac{Y_k \sin(2k-1)\gamma}{(2k-1)\rho^{2k-1}}\right]^{-1} \tag{2.2}$$

$$\gamma = \operatorname{arctg} \frac{h + d(2+h)}{1 + d(2-h)}, \quad d = 2^{-1/2}(1 + 4h^{-2})^{-1/4}$$

$$\rho = [(1 + (2-h)d)^2 + (h + (2+h)d)^2]^{1/2}$$

We determine the stresses on the real axis in both the matrix and the inclusion from the formulas

$$\tau_{xz} = 0, \quad 0 \leqslant x \leqslant \infty \tag{2.3}$$

$$\frac{\tau_{yz}}{\tau_l} = \tau_{yz}^* = \begin{cases} \dfrac{\sqrt{1-x^2}}{2} \displaystyle\sum_{k=1}^{M} \frac{X_k}{2k-1} U_{2(k-1)}(x) + c + 1, & 0 \leqslant x \leqslant 1 \\ 1 - \dfrac{h}{2\beta} \dfrac{1}{\sqrt{x^2-1}} \displaystyle\sum_{k=1}^{M} \frac{X_k}{(x+\sqrt{x^2-1})^{2k-1}}, & x > 1 \end{cases}$$

3. The problem was analysed numerically for $n = 11$. In this case the values of the coefficients $c_i$ are presented in /2, 3/.

The dependence of the stress $\tau_{yz}$ on the coordinate $x$ is given in Fig.1 for the relative stiffness of the inclusion and matrix $\beta = 0.1$ (curves *1* and *3*) and $\beta = 10$ (curve *2*) for values

$h = 0.1$, $0.006$ of the height of the inclusion (curves *1, 2, 3*, respectively). The points denote values of the stresses calculated by means of (2.3) for a constant *c* determined by means of (2.1), and the crosses are values of the same stress but with (2.2) used for *c*. It is seen that in the first case the inclusions (stiffer than the matrix) do not influence the body state of stress, i.e., this constant can be used only for $\mu_0 < \mu$. Moreover, the model of the inclusion /1/ describes the state of stress far from the vertex of the inclusion comparatively well. It should be noted that there is a point on the real axis where the stresses reach the maximum value for the approach described in Sect.1, for $\mu_0 < \mu$ near the vertex of the inclusion in the matrix.

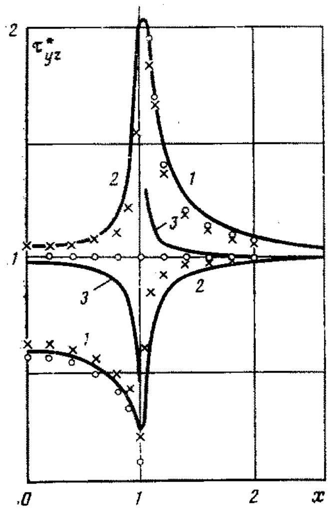


Fig.1

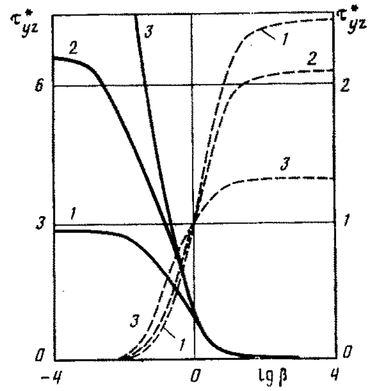


Fig.2

Fig.2 shows the dependence of the stress $\tau_{yz}^*$ on the relative stiffness $\beta$ in the matrix (solid lines) and in the inclusion (dashes) at an endface point for $x = 1$, $y = 0$, and for values of the height of the inclusion $h = 0.1, 0.025, 0.006$ (curves *1, 2, 3*, respectively).

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